Linear Algebra
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3.2 - Algorithm for Linear System

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Learning objectives

After this lecture, you should be able to:

1. apply elimination algorithm and substitution algorithm to solve linear system in two variables;
2. understand the characteristics of linear system which is in triangular form, row echelon form, or reduced row echelon form.
3. detect if a linear system has a unique solution, has no solution, or has an infinitely many solution.

Part 1: Algorithms to solve a system of linear system in **two variables**

1. Elimination algorithm (1)

Given:

$$\begin{cases} L_1: \; x - y = -4 \\ L_2: \; 3x + 2y = 12 \end{cases}$$

Solve the system!

- Multiply the first equation by 2.

$$\begin{cases} 2L_1: \; 2x - 2y = -8 \\ L_2: \; 3x + 2y = 12 \end{cases}$$

- Eliminate the variable y, by adding the two equations.

$$2L_1 + L_2: \; 5x = 4 \;\Leftrightarrow\; x = \frac{4}{5}$$

- Substitute $x = \frac{4}{5}$ back to L_1 or L_2 to find y.

$$x - y = -4 \;\Leftrightarrow\; y = x + 4 = \frac{4}{5} + 4 = \frac{24}{5}$$

1. Elimination algorithm (2)

Assume that the given system has a unique solution.

Input: Non-degenerate linear equations L_1 and L_2 in two variables.

Step 1: *Forward elimination*

- Multiply each equation by a constant s.t. the resulting coefficients of one variable are equal (or negative of the other).
- Subtract (or add) the two equations to eliminate one of the variables.

Step 2: *Back substitution*

- Substitute the value of the variable to an equation of linear system, to obtain the value of the other variable.

2. Substitution algorithm

Given:

$$\begin{cases} L_1: \; x - y = -4 \\ L_2: \; 3x + 2y = 12 \end{cases}$$

Solve the system!

- Represent x in y, in the equation L_1.

$$x = y - 4 \tag{1}$$

- Substitute the value of x in equation (1) to L_2

$$3(y-4) + 2y = 12 \;\Leftrightarrow\; 5y = 24 \;\Leftrightarrow\; y = \frac{24}{5}$$

- Substitute $y = \frac{24}{5}$ to equation (1)

$$x = \frac{24}{5} - 4 = \frac{4}{5}$$

2. Substitution algorithm

Input: Non-degenerate linear equations L_1 and L_2

For simplification, suppose that the variables are x and y.

Step 1:

- Represent one variable, say x, as an equation in y in equation L_1. Then substitute the value of x in L_1 to L_2, to obtain the value of y.

Step 2:

- Substitute the value of the variable y to equation L_1 or L_2, to obtain the value of variable x.

Exercise

Solve the following linear systems using the elimination and substitution algorithms.

1. Solve:

$$\begin{cases} L_1: \ x - 3y = 4 \\ L_2: \ -2x + 6y = 5 \end{cases}$$

2. Solve:

$$\begin{cases} L_1: \ x - 3y = 4 \\ L_2: \ -2x + 6y = -8 \end{cases}$$

Exercise solution (1)

In Exercise 1, we can simplify the second equation, and obtain:

$$\begin{cases} L_1: \; x - 3y = 4 \\ L_2: \; x - 3y = 5 \end{cases}$$

This yields $4 = 5$, which is wrong. So, there is no values of x and y that satisfy the system.

Exercise solution (2)

In Exercise 2, we can simplify the second equation, and obtain:

$$\begin{cases} L_1: \; x - 3y = 4 \\ L_2: \; x - 3y = 4 \end{cases}$$

The two linear equations are equivalent, which means that the lines that represent them coincide (intersect at all points on the coordinate system), and all points on the line satisfy both equations.

How to represent the set of solutions?

$$x - 3y = 4$$

Let $y = t$ for $t \in \mathbb{R}$. Then $x = 3y + 4 = 3t + 4$.

So the set of solutions is $\{x = 3t + 4, y = t, \text{ where } t \in \mathbb{R}\}$

Part 2: Linear systems in triangular and echelon forms

Triangular form

The following system is said to be in triangular form.

$$\begin{cases} 2x_1 - 3x_2 + 5x_3 - 2x_4 = 9 \\ \qquad\quad 5x_2 - \ \ x_3 + 3x_4 = 1 \\ \qquad\qquad\qquad 7x_3 - \ \ x_4 = 3 \\ \qquad\qquad\qquad\qquad\quad 2x_4 = 8 \end{cases} \tag{1}$$

Recall that a **triangular matrix** has one of the following shapes:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

What can you observe?

Triangular form

A system of linear equations is in triangular form if the corresponding coefficient matrix is an upper triangular matrix or a lower triangular matrix, i.e.:

1. The matrix is a square matrix;
2. The entries below the main diagonal (resp. above the main diagonal, for the upper triangular matrix) are 0;

Remark:

- We do not care of the values in the main diagonals (they can be 0)

Solving system in (*upper*) triangular form

$$\begin{cases} 2x_1 - 3x_2 + 5x_3 - 2x_4 = 9 \\ \phantom{2x_1 - {}} 5x_2 - x_3 + 3x_4 = 1 \\ \phantom{2x_1 - 3x_2 + {}} 7x_3 - x_4 = 3 \\ \phantom{2x_1 - 3x_2 + 5x_3 - {}} 2x_4 = 8 \end{cases} \tag{2}$$

Algorithm to solve the system:

1. Solve the last equation to get x_4;
2. Substitute x_4 to the third equation to obtain x_3;
3. Substitute x_3 and x_4 to the second equation to obtain x_2;
4. Substitute x_2, x_3, and x_4 to the first equation to obtain x_1.

Exercise: Find the solution of the system!

Solution of the exercise

- From the last equation, we get: $x_4 = 4$
- From the third equation:

$$x_3 = \frac{x_4 + 3}{7} = \frac{4 + 3}{7} = 1$$

- From the second equation:

$$x_2 = \frac{x_3 - 3x_4 + 1}{5} = \frac{1 - 3(4) + 1}{5} = \frac{-10}{5} = -2$$

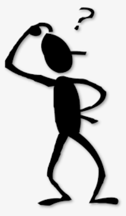
- From the first equation:

$$\begin{aligned} x_1 &= \frac{3x_2 - 5x_3 + 2x_4 + 9}{2} = \frac{3(-2) - 5(1) + 2(4) + 9}{2} \\ &= \frac{-6 - 5 + 8 + 9}{2} = \frac{6}{2} = 3 \end{aligned}$$

So, the solution is: $x_1 = 3, x_2 = -2, x_3 = 1, x_4 = 4$

Echelon form

Now, what if the coefficient matrix is not a square matrix ???



Echelon form

$$\begin{cases} \mathbf{2}x_1 - 4x_2 + 5x_3 - 2x_4 + x_5 = 9 \\ \qquad\quad \mathbf{2}x_2 - 2x_3 + 4x_4 - 2x_5 = 1 \\ \qquad\qquad\qquad \mathbf{1}x_3 - x_4 = 3 \end{cases}$$

The system is said to be in echelon form, that is:

1. All rows consisting of **only zeroes** are at the **bottom**.
2. The leading coefficient (also called the pivot) of a nonzero row is always **strictly to the right** of the leading coefficient of the row above it.
3. (In many literature), the pivot is 1 (this is called *leading one*).

*

Characteristics:

- The leading variables (x_1, x_2, x_3) in the system are called pivot;
- The other variables (x_4 and x_5) are free variables.

*Note that in Howard Anton's book, the main coefficient is always 1, which is different from the definition in this lecture module.

Echelon form (general form)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \cdots + a_{1n}x_n &= b_1 \\
a_{2j_2}x_{j_2} + a_{2j_2}x_{j_2+1} + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
a_{rj_r}x_{j_r} + \cdots + a_{rn}x_n &= b_r
\end{aligned}
$$

where $1 < j_2 < \cdots < j_r$ and $a_{11}, a_{2j_2}, \ldots, a_{rj_r} \neq 0$.

The **pivot** variables are: $x_1, x_{j_2}, \ldots, x_{j_r}$

Remark: in order for the system to have a solution, it must be $r \leq n$.

Then...is there any difference between triangular form and echelon form?

Clarification

The following matrix **is echelon, but not triangular**

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

The following matrix **is triangular, but not echelon**

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The following matrix **is echelon and triangular** (LEFT), and **not echelon and not triangular** (RIGHT)

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

Remark. For non-singular square matrices, "row echelon" and "upper triangular" are equivalent.

Part 3: How to determine the number of solutions?

How to know the number of solutions?

Given a system of linear equations with r equations in n variables.

Determine the conditions such that:

- the system has a **unique solution**?
- the system has **no solution**?
- the system has an **infinite number of solutions**?

How to know the number of solutions?

Given a system of linear equations with r equations in n variables.

Then:

- the system has a **unique solution**
  - when $r = n$ (where no equation is a linear combination of another)
- the system has **no solution**
  - when $r > n$, and no equation is a linear combination of another
- the system has an **infinite number of solutions**
  - when $r < n$

How to write the solutions when there are infinitely many? *(case when $r < n$)*

Given:

$$\begin{cases} x_1 - 4x_2 + 5x_3 - 2x_4 + x_5 = 9 \\ x_2 - 2x_3 + 4x_4 - 2x_5 = 1 \\ x_3 - x_4 = 3 \end{cases}$$

- Pivot variables: x_1, x_2, x_3
- Free variables: x_4, x_5

Algorithm to solve the system:

1. Assign *parameters* to the free variables;

$$x_4 = a \quad \text{and} \quad x_5 = b$$

2. Substitute the variables back to obtain the value of pivot variables.

1. Solution in parametric form

- From the third equation:

$$x_3 = x_4 + 3 = a + 3$$

- From the second equation:

$$\begin{aligned} x_2 &= 2x_3 - 3x_4 + 2x_5 + 1 \\ &= 2(a+3) - 4a + 2b + 1 = -2a + 2b + 7 \end{aligned}$$

- From the first equation:

$$\begin{aligned} x_1 &= 3x_2 - 5x_3 + 2x_4 - x_5 + 9 \\ &= 3(-2a + 2b + 7) - 5(a+3) + 2a - b + 9 \\ &= -9a + 5b + 15 \end{aligned}$$

Set of solutions:

$$\{-9a + 5b + 15, -2a + 2b + 7, a + 3, a, b\}$$

2. Solution in free-variable form

Use back-substitution to solve the system, and obtain the pivot variables.

$$\begin{cases} x_1 & = 4x_2 - 5x_3 + 2_4 - x_5 - 9 \\ x_2 & = 2x_3 - 4x_4 + 2x_5 + 1 \\ x_3 & = x_4 + 3 \\ x_4 & = \text{free variable} \\ x_5 & = \text{free variable} \end{cases}$$

Set of solutions:

$$\{(4x_2 - 5x_3 + 2_4 - x_5 - 9),\ (2x_3 - 4x_4 + 2x_5 + 1),\ (x_4 + 3),\ x_4,\ x_5\}$$

Part 4: The reduced row echelon form

The reduced row echelon form

A matrix is in reduced row echelon form (also called row canonical form), if it satisfies the following conditions are satisfied:

1. It is in **row echelon form**.
2. The leading entry in each nonzero row is a 1 (called a leading 1).
3. Each **column** containing a leading 1 has zeros in **all** its other entries.

Which matrix is in a reduced row echelon form?

- $A = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
- $B = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
- $C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

How to transform the coefficient matrix into a triangular or (reduced) row echelon form?

Apply the **elementary row operations**.

In the next lecture, we will learn

how to solve a linear system by transforming the coefficient matrix to a reduced row echelon form.

to be continued...